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SEPARATION OF CONTACTING SURFACES IN THERMOELASTIC INTERACTION OF TWO CYLINDERS WITH TIME-DEPENDENT HEAT RELEASE DUE TO FRICTION

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The problem of a thermoelastic interaction of two cylinders with separation of their contact surfaces due to local loading of lateral surfaces is formulated and solved. The effect of the multiply connected contact region is shown to exist under a certain relation between the linear thermal expansion coefficients of the bodies.

Key words: *contact interaction, cylinder, heat release due to friction, nonstationary temperature, separation of contact surfaces.*

The contact problem of two interacting hollow cylinders tightly inserted one into the other and compressed by a load varied along the tribosystem axis was considered in [1] in the axisymmetric formulation. Investigations into the solution of this problem performed under the assumption of an intimate contact of the cylinders over their entire surface proved that a load localized within a certain interval can change the sign of the contact stress and, therefore, give rise to separation zones whose size increases with increasing heat-release intensity, provided that the thermal expansion coefficient of the inner cylinder is smaller compared to that of the outer cylinder. As the thermal expansion coefficient of the outer cylinder decreases, simple connectedness of the load-application region no longer guarantees simple connectedness of the loaded contact region.

Mathematical Formulation of the Problem and Solution Construction. The condition of existence of separation zones requires the following modification of the problem formulation proposed in [1]: since a loaded contact zone, a separation zone, and an unloaded contact zone can be identified on the contact surface, individual thermophysical conditions must be set in each of these zones. In particular, the following conditions must be fulfilled at $r = a_0$:

— in the loaded contact zone, heat-release condition and the condition of an imperfect thermal contact

$$\lambda_1 \partial_r T_1 - \lambda_2 \partial_r T_2 = f\omega(\tau) a_0 p(z, \tau); \qquad (1)$$

$$\lambda_1 \partial_r T_1 + \lambda_2 \partial_r T_2 + h_a(T_1 - T_2) = 0; \qquad (2)$$

— in the unloaded contact zone, conditions of an imperfect thermal contact with another value of the thermal conductivity coefficient

$$\lambda_1 \partial_r T_1 = \lambda_2 \partial_r T_2 = -h_c(T_1 - T_2); \qquad (3)$$

— in the separation zone, depending on the chosen model, either the heat insulation conditions on the surfaces

$$\partial_r T_1 = \partial_r T_2 = 0, \qquad (4)$$

or the conditions of an imperfect thermal contact through an interlayer

$$\lambda_1 \partial_r T_1 = \lambda_2 \partial_r T_2 = -h_b(T_1 - T_2). \qquad (5)$$

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The mechanical conditions at the surface $r = a_0$ in the unloaded contact zone are the postulated equality of the radial stresses and displacements and the condition of zero shear stress

$$\sigma_r^{(1)} = \sigma_r^{(2)} = -p(z,\tau), \qquad u_1 = u_2, \qquad \tau_{rz}^{(1)} = \tau_{rz}^{(2)} = 0. \tag{6}$$

In the separation zone and in the unloaded contact zone, conditions of zero radial and shear stresses are applied:

$$\sigma_r^{(1)} = \sigma_r^{(2)} = 0, \qquad \tau_{rz}^{(1)} = \tau_{rz}^{(2)} = 0. \tag{7}$$

Here, the boundaries between the zones are initially unknown and should be found during the solution construction.

Mathematically, the problem reduces to integration of a system that includes the following differential equations:

— heat-conduction equation

$$\partial_r^2 T_j + r^{-1}\partial_r T_j + \partial_z^2 T_j = k_j^{-1}\partial_\tau T_j;$$

— equation of equilibrium

$$\partial_r \sigma_r^{(j)} + r^{-1}(\sigma_r^{(j)} - \sigma_\theta^{(j)}) + \partial_z \tau_{rz}^{(j)} = 0, \quad \partial_r \tau_{rz}^{(j)} + r^{-1}\tau_{rz}^{(j)} + \partial_z \sigma_z^{(j)} = 0;$$

— equation of strain consistency

$$\partial_r \varepsilon_\theta^{(j)} + r^{-1}(\varepsilon_\theta^{(j)} - \varepsilon_r^{(j)}) = 0, \qquad r\partial_z^2 \varepsilon_\theta^{(j)} + \partial_r \varepsilon_z^{(j)} = \partial_z \gamma_{rz}^{(j)};$$

— the Hooke's law relations

$$E_j \varepsilon_r^{(j)} = \sigma_r^{(j)} - \nu_j(\sigma_\theta^{(j)} + \sigma_z^{(j)}) + E_j\alpha_j T_j, \qquad E_j \varepsilon_\theta^{(j)} = \sigma_\theta^{(j)} - \nu_j(\sigma_r^{(j)} + \sigma_z^{(j)}) + E_j\alpha_j T_j,$$

$$E_j \varepsilon_z^{(j)} = \sigma_z^{(j)} - \nu_j(\sigma_r^{(j)} + \sigma_\theta^{(j)}) + E_j\alpha_j T_j, \qquad E_j \gamma_{rz}^{(j)} = 2(1+\nu_j)\tau_{rz}^{(j)} \qquad (j = 1, 2).$$

The initial conditions are

$$T_j(r, z, 0) = 0;$$

the boundary conditions are

$$r = a_1: \qquad \partial_r T_1 = \gamma_1 T_1, \quad \sigma_r^{(1)} = -q_1(z,\tau), \quad \tau_{rz}^{(1)} = 0,$$

$$r = a_2: \qquad \partial_r T_2 = -\gamma_2 T_2, \quad \sigma_r^{(2)} = -q_2(z,\tau), \quad \tau_{rz}^{(2)} = 0$$

and contact conditions are (1)–(7).

Hereinafter, r and z are the radial and axial coordinates, τ is the time, $p(z,\tau)$ is the contact pressure, $q_j(z,\tau)$ is the external load on noncontacting surfaces of the tribosystem, $\omega(\tau)$ is the relative angular velocity of revolution, T_j is the temperature, $\sigma_r^{(j)}$, $\sigma_\theta^{(j)}$, and $\sigma_z^{(j)}$ are the radial, circumferential, and axial normal pressures, respectively, $\tau_{rz}^{(j)}$ is the shear stress, $\varepsilon_r^{(j)}$, $\varepsilon_\theta^{(j)}$, and $\varepsilon_z^{(j)}$ are the radial, tangent, and axial linear strains, respectively, $\gamma_{rz}^{(j)}$ is the shear strain, $u_r^{(j)}$ is the radial displacement, E_j is Young's modulus, ν_j, λ_j, k_j, and α_j are Poisson's ratio, the thermal conductivity, thermal diffusivity, and linear expansion coefficient, respectively, $\gamma_j = \bar{\alpha}_j/\lambda_j$, $\bar{\alpha}_j$ is the heat-transfer coefficient, f is the friction coefficient, h_a, h_b, and h_c are the thermal conductivities of the three zones on the contact surface, $r = a_1$ and $r = a_2$ are the outer surfaces of the inner and outer cylinders, respectively, and $r = a_0$ is the contact surface in the unloaded system; $j = 1$ and $j = 2$ refer to the inner and outer cylinders, respectively.

We reduce the posed problem to a system of integral equations for the contact pressure $p(z,\tau)$ and two functions $f_j(z,\tau)$ $(j = 1, 2)$ proportional to the heat fluxes at the contact surface:

$$f_j(z,\tau) = (-1)^{j-1}\partial_r T_j(a_0, z, \tau).$$

Using relations (2.2), (2.4), (2.7), and (2.11) from [1], we write the following expression for the temperature of the cylinders:

$$T_j(r,z,\tau) = \frac{1}{\pi}\,\partial_\tau \int_0^\tau \int_{-\infty}^{\infty} f_j(t,\eta)\,\Phi_j(r, t - z, \tau - \eta)\,dt\,d\eta.$$

Here

$$\Phi_j(r,z,\tau) = \int\limits_0^\infty \bar\Phi_{j,st}(r,\xi)\cos(\xi z)\,d\xi \mp \frac{\pi}{2}\,a_0 \sum_{m=1}^\infty \frac{W_0(\mu_{j,m}r,\mu_{j,m}a_0)W_0(\mu_{j,m}a_0,\mu_{j,m}a_0)}{\mu_{j,m}\,N_{j,m}^2}$$

$$\times \sum_{k=1}^2 \exp\left((-1)^k\mu_{j,m}z\right)\,\mathrm{erfc}\left(\mu_{j,m}\sqrt{k_j\tau} + (-1)^k\,\frac{z}{2\sqrt{k_j\tau}}\right);$$

$$\bar\Phi_{j,st}(r,\xi) = \pm\frac{1}{\xi}\,\frac{I_0(\xi r)(\xi K_1(\xi a_j) \pm \gamma_j K_0(\xi a_j)) + K_0(\xi r)(\xi I_1(\xi a_j) \mp \gamma_j I_0(\xi a_j))}{I_1(\xi a_0)(\xi K_1(\xi a_j) \pm \gamma_j K_0(\xi a_j)) - K_1(\xi a_0)(\xi I_1(\xi a_j) \mp \gamma_j I_0(\xi a_j))};$$

$$W_0(x,y) = J_0(x)Y_1(y) - Y_0(x)J_1(y); \qquad W_1(x,y) = J_1(x)Y_1(y) - Y_1(x)J_1(y);$$

$$N_{j,m}^2 = a_0^2W_0^2(\mu_{j,m}a_0,\mu_{j,m}a_0) - a_j^2(1+\gamma_j^2\,\mu_{j,m}^{-2})W_0^2(\mu_{j,m}a_j,\mu_{j,m}a_0).$$

Here, $I_\nu(z)$ and $K_\nu(z)$ are the modified Bessel functions of the first and second kind of order ν, $J_\nu(z)$ and $Y_\nu(z)$ are the Bessel functions of the first and second kind of order ν, and $\mathrm{erfc}\,(z)$ is the error function [2]. The eigenvalues $\mu_{j,m}$ are the roots of the transcendental equation

$$\mu_j W_1(\mu_j a_j,\mu_j a_0) \pm \gamma_j W_0(\mu_j a_j,\mu_j a_0) = 0.$$

The upper and lower signs in the combinations $\pm$ and $\mp$ refer to the inner cylinder ($j=1$) and to the outer cylinder ($j=2$), respectively.

We invert the Fourier transform of the integral representation of radial displacements on the contact surface of the cylinders (see relation (2.15) in [1]) obtained with stresses set at the boundaries of the cylinders; for $r=a_0$, we write the following expression:

$$u_r^{(j)}(a_0,z,\tau) = \frac{1-\nu_j^2}{E_j}\Big(\frac{a_0}{\pi}\int\limits_{-\infty}^\infty p(t,\tau)\,\Delta_1(a_j,t-z)\,dt$$

$$-\frac{a_j}{2\pi}\int\limits_{-\infty}^\infty\int\limits_{-\infty}^\infty q_j(t,\tau)\,\exp\left(i\xi(t-z)\right)\bar\Delta_2(a_j,\xi)\,dt\,d\xi\Big) + \frac{\alpha_j}{\pi}\,\partial_\tau\int\limits_0^\tau\int\limits_{-\infty}^\infty f_j(t,\eta)H_j(t-z,\tau-\eta)\,dt\,d\eta.$$

Here,

$$\Delta_1(a_j,z) = \int\limits_0^\infty \bar\Delta_1(a_j,\xi)\,\cos(\xi z)\,d\xi; \qquad H_j(z,\tau) = \int\limits_0^\infty(\bar H_{j,\mathrm{st}}(\xi) + \bar H_{j,0}(\xi,\tau))\cos(\xi z)\,d\xi;$$

$$\bar H_{j,\mathrm{st}}(\xi) = (1-\nu_j^2)\xi^{-2}[\bar\Delta_2(a_j,\xi)\partial_r\bar\Phi_{j,\mathrm{st}}(a_j,\xi) \mp (\bar\Delta_1(a_j,\xi) - (1-\nu_j)^{-1})];$$

$$\bar H_{j,0}(\xi,\tau) = \pm 2(1+\nu_j)a_0\sum_{m=1}^\infty \frac{W_0(\mu_{j,m}a_0,\mu_{j,m}a_0)}{N_{j,m}^2(\xi^2+\mu_{j,m}^2)^2}\Big[\xi^2(\bar\Delta_1(a_j,\xi)a_0W_0(\mu_{j,m}a_0,\mu_{j,m}a_0)$$

$$-\bar\Delta_2(a_j,\xi)a_jW_0(\mu_{j,m}a_j,\mu_{j,m}a_0)) + \bar\Delta_3(a_j,\xi)\mu_{j,m}W_1(\mu_{j,m}a_j,\mu_{j,m}a_0)\Big]\exp\left(-k_j(\xi^2+\mu_{j,m}^2)\tau\right);$$

$$\bar\Delta_j(a_j,\xi) = \tilde\Delta_j(a_j,\xi)\tilde\Delta_0^{-1}(a_j,\xi);$$

$$\tilde\Delta_0(a_j,\xi) = 4(1-\nu_j) + a_j^2\xi^2 + a_0^2\xi^2 + (2(1-\nu_j)+a_j^2\xi^2)(2(1-\nu_j)+a_0^2\xi^2)$$

$$\times[I_1(a_j\xi)K_1(a_0\xi) - I_1(a_0\xi)K_1(a_j\xi)]^2 - a_j^2\xi^2(2(1-\nu_j)+a_0^2\xi^2)$$

$$\times[I_0(a_j\xi)K_1(a_0\xi) + I_1(a_0\xi)K_0(a_j\xi)]^2 - a_0^2\xi^2(2(1-\nu_j)+a_j^2\xi^2)$$

$$\times[I_1(a_j\xi)K_0(a_0\xi) + I_0(a_0\xi)K_1(a_j\xi)]^2 + a_j^2a_0^2\xi^4[I_0(a_j\xi)K_0(a_0\xi) - I_0(a_0\xi)K_0(a_j\xi)]^2;$$

$$\tilde{\Delta}_1(a_j,\xi) = 2\Big[1 + (2(1-\nu_j) + a_j^2\xi^2)[I_1(a_j\xi)K_1(a_0\xi) - I_1(a_0\xi)K_1(a_j\xi)]^2 - a_j^2\xi^2[I_0(a_j\xi)K_1(a_0\xi) + I_1(a_0\xi)K_0(a_j\xi)]^2\Big];$$

$$\tilde{\Delta}_2(a_j,\xi) = 2a_0\xi[I_1(a_j\xi)K_0(a_0\xi) + I_0(a_0\xi)K_1(a_j\xi)] - 2a_j\xi[I_0(a_j\xi)K_1(a_0\xi) + I_1(a_0\xi)K_0(a_j\xi)];$$

$$\tilde{\Delta}_3(a_j,\xi) = 2\Big[(2(1-\nu_j) + a_j^2\xi^2)[I_1(a_j\xi)K_1(a_0\xi) - I_1(a_0\xi)K_1(a_j\xi)] - a_j a_0\xi^2[I_0(a_j\xi)K_0(a_0\xi) - I_0(a_0\xi)K_0(a_j\xi)]\Big].$$

To find the unknown contact pressure $p(z,\tau)$ and the functions $f_j(z,\tau)$, we use the thermophysical contact conditions (1)–(5) and the condition of equality for displacements in (6); with these conditions fulfilled, we obtain the following systems of equations:

— in the loaded contact zone,

$$\lambda_1 f_1(z,\tau) + \lambda_2 f_2(z,\tau) = f\omega(\tau)a_0 p(z,\tau),$$

$$\sum_{k=1}^{2}(-1)^{k-1}\Big[\lambda_k f_k(z,\tau) + \frac{h_a}{\pi}\partial_\tau\int_0^\tau\int_{-\infty}^{\infty} f_k(t,\eta)\,\Phi_k(a_0,t-z,\tau-\eta)\,dt\,d\eta\Big] = 0,$$

$$\frac{a_0E_0}{\pi}\int_{-\infty}^{\infty} p(t,\tau)\Big[\sum_{k=1}^{2}(-1)^k\frac{1-\nu_k^2}{E_k}\Delta_1(a_k,t-z)\Big]\,dt + \sum_{k=1}^{2}(-1)^k\frac{\alpha_k E_0}{\pi}\,\partial_\tau\int_0^\tau\int_{-\infty}^{\infty} f_k(t,\eta)H_k(t-z,\tau-\eta)\,dt\,d\eta$$

$$= \sum_{k=1}^{2}(-1)^k\frac{1-\nu_k^2}{E_k}\,\frac{a_kE_0}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} q_j(t,\tau)\,\exp\left(i\xi(t-z)\right)\bar{\Delta}_2(a_j,\xi)\,dt\,d\xi,$$

where $E_0 = (2((1-\nu_1^2)/E_1 + (1-\nu_2^2)/E_2))^{-1}$;

— in the unloaded contact zone,

$$\lambda_j f_j(z,\tau) + h_c\sum_{k=1}^{2}(-1)^{k+j}\frac{1}{\pi}\partial_\tau\int_0^\tau\int_{-\infty}^{\infty} f_k(t,\eta)\,\Phi_k(a_0,t-z,\tau-\eta)\,dt\,d\eta = 0 \quad (j=1,2), \qquad p(z,\tau)=0; \tag{8}$$

— in the separation zone, depending on the chosen model or condition either the condition

$$f_1(z,\tau) = f_2(z,\tau) = p(z,\tau) = 0,$$

or conditions of type (8) with h_c replaced by h_b. Here, as was noted above, the boundaries between the contact zones are initially unknown.

Definition and Construction of the Numerical Algorithm. To construct the solution of such a system of equations, we propose a numerical algorithm that involves the results of [1] and some specific properties of the above-obtained functions (it can be stated that $\Phi_j(r,z,0)=0$ and $H_j(z,0)=0$).

We divide the time interval $[0,\tau_*]$ in which the behavior of the tribosystem is examined into N segments by choosing the times $\tau_i = i\tau_1$ $(i=0,\ldots,N)$, where $\tau_N=\tau_*$, and perform time discretization of the integrals

$$F(z,\tau) = \frac{1}{\pi}\,\partial_\tau\int_0^\tau\int_{-\infty}^{\infty} f(t,\eta)\,\Phi(t-z,\tau-\eta)\,dt\,d\eta \qquad [\Phi(z,0)=0]$$

by the scheme

$$F(z,0)=0, \qquad F(z,\tau_1) = 0.5G(z,\tau_{1,1}) + 0.25G(z,\tau_{0,2}),$$

$$F(z,\tau_2) = 0.5G(z,\tau_{2,1}) + 0.5G(z,\tau_{1,2}) + 0.25(G(z,\tau_{0,3}) - G(z,\tau_{0,1})),$$

$$F(z,\tau_n) = 0.5G(z,\tau_{n,1}) + 0.5G(z,\tau_{n-1,2})$$

$$+\,0.5\sum_{k=1}^{n-2}(G(z,\tau_{k,n+1-k}) - G(z,\tau_{k,n-1-k})) + 0.25(G(z,\tau_{0,n+1}) - G(z,\tau_{0,n-1})) \qquad (n\geqslant 3).$$

Here

$$G(z,\tau_{i,j}) = \frac{1}{\pi}\int\limits_{-\infty}^{\infty} f(t,\tau_i)\Phi(t-z,\tau_j)\,dt.$$

Then, we obtain the following systems of equations for each time τ_i $(i = 0,\dots,N)$:
— in the loaded contact zone,

$$\lambda_1 f_1(z,\tau_i) + \lambda_2 f_2(z,\tau_i) = f\omega(\tau_i)a_0 p(z,\tau_i),$$

$$\sum_{k=1}^{2}(-1)^{k-1}\Big[\lambda_k f_k(z,\tau_i) + \frac{h_a}{2\pi}\int\limits_{-\infty}^{\infty} f_k(t,\tau_i)\,\Phi_k(a_0,t-z,\tau_1)\,dt\Big] = h_a\sum_{k=1}^{2}(-1)^k R'_k(a_0,z,\tau_i), \tag{9}$$

$$\frac{a_0E_0}{\pi}\int\limits_{-\infty}^{\infty} p(t,\tau_i)\Big[\sum_{k=1}^{2}(-1)^k\frac{1-\nu_k^2}{E_k}\,\Delta_1(a_k,t-z)\Big]\,dt$$

$$+\sum_{k=1}^{2}(-1)^k\frac{\alpha_kE_0}{2\pi}\int\limits_{-\infty}^{\infty} f_k(t,\tau_i)H_k(t-z,\tau_1)\,dt = \sum_{k=1}^{2}(-1)^k\frac{1-\nu_k^2}{E_k}\,\frac{a_kE_0}{\pi}$$

$$\times\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} q_k(t,\tau_i)\,\bar\Delta_2(a_k,\xi)\exp\left(i\xi(t-z)\right)dt\,d\xi + E_0\sum_{k=1}^{2}(-1)^{k-1}\alpha_kR''_k(z,\tau_i);$$

— in the unloaded contact zone,

$$\begin{gathered}\lambda_j f_j(z,\tau_i) + h_c\sum_{k=1}^{2}(-1)^{k+j}\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} f_k(t,\tau_i)\,\Phi_k(a_0,t-z,\tau_1)\,dt = h_c\sum_{k=1}^{2}(-1)^{k+j-1}R'_k(a_0,z,\tau_i),\\ p(z,\tau_i) = 0;\end{gathered} \tag{10}$$

— in the separation zone,

$$f_1(z,\tau_i) = f_2(z,\tau_i) = p(z,\tau_i) = 0. \tag{11}$$

Here

$$R'_k(r,z,0) = 0; \qquad R'_k(r,z,\tau_1) = 0.25G'_k(r,z,\tau_{0,2});$$

$$R'_k(r,z,\tau_2) = 0.5G'_k(r,z,\tau_{1,2}) + 0.25(G'_k(r,z,\tau_{0,3}) - G'_k(r,z,\tau_{0,1}));$$

$$R'_k(r,z,\tau_n) = 0.5G'_k(r,z,\tau_{n-1,2}) + 0.5\sum_{\ell=1}^{n-2}\Big(G'_k(r,z,\tau_{\ell,n+1-\ell}) - G'_k(r,z,\tau_{\ell,n-1-\ell})\Big)$$

$$+0.25\Big(G'_k(r,z,\tau_{0,n+1}) - G'_k(r,z,\tau_{0,n-1})\Big) \quad (n \geqslant 3);$$

$$G'_k(r,z,\tau_{i,j}) = \frac{1}{\pi}\int\limits_{-\infty}^{\infty} f_k(t,\tau_i)\,\Phi_k(r,t-z,\tau_j)\,dt;$$

the function R''_k is defined similarly to R'_k if

$$G''_k(z,\tau_{i,j}) = \frac{1}{\pi}\int\limits_{-\infty}^{\infty} f_k(t,\tau_i)H_k(t-z,\tau_j)\,dt.$$

For the temperature, we have the relation

$$T_j(r,z,\tau_i) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} f_j(t,\tau_i)\Phi_j(r,t-z,\tau_1)\,dt + R'_j(r,z,\tau_i).$$

We consider the properties of the kernels $\Delta(a_j, z)$, $H_j(z, \tau)$, and $\Phi_j(r, z, \tau)$. Since

$$\bar{\Delta}_1(a_j, 0) = (1 - \nu_j^2)^{-1}\Big(\frac{a_j^2 + a_0^2}{a_j^2 - a_0^2} + \nu_j\Big),$$

$$\bar{\Phi}_j(r, 0, \tau) = a_0(\pm \ln(r/a_j) + (a_j\gamma_j)^{-1})$$

$$\mp 2a_0 \sum_{m=1}^{\infty} \frac{W_0(\mu_{j,m} r, \mu_{j,m} a_0) W_0(\mu_{j,m} a_0, \mu_{j,m} a_0)}{\mu_{j,m}^2 N_{j,m}^2} \exp(-k_j \mu_{j,m}^2 \tau),$$

$$\bar{H}_j(0, \tau) = \pm a_0^2 \Big[\frac{a_0^2}{a_0^2 - a_j^2} \ln\Big(\frac{a_0}{a_j}\Big) - 0.5 \pm \frac{1}{a_j\gamma_j}$$

$$+ \frac{4a_j}{a_0^2 - a_j^2} \sum_{m=1}^{\infty} \frac{W_1(\mu_{j,m} a_j, \mu_{j,m} a_0) W_0(\mu_{j,m} a_0, \mu_{j,m} a_0)}{N_{j,m}^2 \mu_{j,m}^3} \exp(-k_j \mu_{j,m}^2 \tau)\Big],$$

and, as $\xi \to \infty$,

$$\bar{\Delta}_1(a_j, \xi) \approx \mp 2(a_0\xi)^{-1}, \qquad \bar{\Phi}_j(r, \xi, \tau) \approx \bar{\Phi}_{j,\mathrm{st}}(r, \xi) \approx \xi^{-1}\sqrt{a_0/r}\, \exp(\mp\xi(a_0 - r)),$$

$$\bar{H}_j(\xi, \tau) \approx \bar{H}_{j,\mathrm{st}}(\xi) \approx \pm(1 + \nu_j)/\xi^2 \qquad (\tau > 0),$$

then, based on the results obtained in [3], we can argue that the kernels

$$\Delta(z) = \int_0^{\infty} \bar{\Delta}(\xi) \cos(\xi z)\, d\xi = a_0 E_0 \int_0^{\infty} \Big[\sum_{k=1}^{2} (-1)^k \frac{1 - \nu_k^2}{E_k} \bar{\Delta}_1(a_k, \xi)\Big] \cos(\xi z)\, d\xi$$

and $\Phi_{j,\mathrm{st}}(a_0, z)$ have a logarithmic singularity, whereas the kernels $H_j(\xi, \tau)$ and $\Phi_{j,\mathrm{st}}(r, z)$ (at $r \neq a_0$) are regular. Then,

$$\Delta(z) = K_0(|z|) + \int_0^{\lambda_1} \bar{\Delta}(\xi) \cos(\xi z)\, d\xi - \ln(2\lambda_1) \qquad (z = 0),$$

$$\Delta(z) = K_0(|z|) + \int_0^{\lambda_1} \bar{\Delta}(\xi) \cos(\xi z)\, d\xi - \mathrm{Ci}(\lambda_1 |z|) - K_0(|z|) \qquad (z \neq 0),$$

$$H_j(z, \tau) = \int_0^{\lambda_{j,2}} \bar{H}_j(\xi, \tau) \cos(\xi z)\, d\xi \pm (1 + \nu_j)\Big[\frac{\cos(\lambda_{j,2} z)}{\lambda_{j,2}} + |z|\Big(\mathrm{Si}(\lambda_{j,2}|z|) - \frac{\pi}{2}\Big)\Big],$$

$$\Phi_{j,\mathrm{st}}(a_0, z) = K_0(|z|) + \int_0^{\lambda_{j,3}} \bar{\Phi}_{j,\mathrm{st}}(a_0, \xi) \cos(\xi z)\, d\xi - \ln(2\lambda_{j,3}) \qquad (z = 0),$$

$$\Phi_{j,\mathrm{st}}(a_0, z) = K_0(|z|) + \int_0^{\lambda_{j,3}} \bar{\Phi}_{j,\mathrm{st}}(a_0, \xi) \cos(\xi z)\, d\xi - \mathrm{Ci}(\lambda_{j,3}|z|) - K_0(|z|) \qquad (z \neq 0),$$

$$\Phi_{j,\mathrm{st}}(r, z) = \int_0^{\lambda_{j,3}} \bar{\Phi}_{j,\mathrm{st}}(r, \xi) \cos(\xi z)\, d\xi + \frac{1}{2}\sqrt{\frac{a_0}{r}} \sum_{k=1}^{2} E_1(\pm\lambda_{j,3}(a_0 - r + (-1)^{k-1} i z)) \qquad (r \neq a_0),$$

where $i^2 = -1$, $\mathrm{Si}(z)$ and $\mathrm{Ci}(z)$ are the integral sine and cosine; $E_1(z)$ is the integral exponential [2]. We take into account that $K_0(z) \approx \ln(2/z) - \gamma$ as $z \to 0$ (γ is the Euler constant) [2].

The integration boundaries $\lambda_{j,k}$ were chosen so that the integrands in the Fourier integrals over the intervals $(\lambda_{j,k}, \infty)$ could be replaced by their asymptotic expressions. We find the values of the integrals over the intervals $[0, \lambda_{j,k}]$ by numerical integration, using the Filon method [4].

We represent the right side of the third integral equation in (9) as

$$\sum_{k=1}^{2}(-1)^k \frac{1-\nu_k^2}{E_k}\,\frac{a_k E_0}{\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} q_k(t,\tau)\,\bar{\Delta}_2(a_k,\xi)\exp\left(i\xi(t-z)\right)dt\,d\xi$$

$$= \frac{1-\nu_2^2}{E_2}\,\frac{2a_2E_0}{\pi}\,q(\tau)(I_0(L+z)+I_0(L-z)),$$

$$I_0(z) = \int_0^{\lambda_4} \frac{\bar{\Delta}_2(a_2,\xi)-\bar{\Delta}_2(a_2,0)}{\xi}\,\sin\left(\xi z\right)d\xi$$

$$+\,\bar{\Delta}_2(a_2,0)\,\mathrm{Si}\left(\lambda_4 z\right) + \frac{2(a_2-a_0)}{i\sqrt{a_2a_0}}\sum_{k=1}^{2}(-1)^{k-1}E_1(\lambda_4(a_2-a_0+(-1)^k iz)).$$

The inner surface of the tribosystem is free of stresses, and the load on the outer surface is a function symmetric about the plane $z=0$, which varies according to the law

$$q_1(z,\tau)=0, \qquad q_2(z,\tau)=q(\tau)H(L-|z|)$$

[$H(z)$ is the Heaviside function]. Here, we take into account that the kernel $\bar{\Delta}_2(a_2,\xi)$ has the following properties:

$$\bar{\Delta}_2(a_2,0)=\frac{1}{1-\nu_2^2}\,\frac{2a_0a_2}{a_2^2-a_0^2}, \qquad \bar{\Delta}_2(a_2,\xi)|_{\xi\to\infty}\approx\frac{4(a_2-a_0)}{\sqrt{a_2a_0}}\,\exp\left(-\xi(a_2-a_0)\right).$$

An analysis of system (9)–(11) shows that $p(z,\tau_k)$ at any time τ_k ($k=0,\dots,N$) is a bounded and continuous function and $f_j(z,\tau_k)$ is a bounded function that has a limited number of local extrema and first-kind discontinuity points; in other words, these functions satisfy the Dirichlet conditions [5]. We take into account that the loading symmetry guaranties the symmetry of the solution and, in particular, the symmetry of the contact pressure and the functions $f_j(z,\tau_k)$ relative to the plane $z=0$; then, we represent these functions in the domain $z\in[0,\infty)$ as truncated series in generalized Laguerre polynomials [6]:

$$p(z,\tau_k)=\frac{\exp\left(-z\right)}{\sqrt{z}}\sum_{m=0}^{M}B_m(\tau_k)L_m^{(-1/2)}(2z),$$

$$f_j(z,\tau_k)=\frac{\exp\left(-z\right)}{\sqrt{z}}\sum_{m=0}^{M}C_{j,m}(\tau_k)L_m^{(-1/2)}(2z). \tag{12}$$

We substitute expressions (12) into system (9)–(11) to subsequently pass from integrals over the interval $(-\infty,\infty)$ to integrals over the interval $[0,\infty)$, using the following scheme [$\psi(-z,\tau)=\psi(z,\tau)$]:

$$\int_{-\infty}^{\infty}\psi(t,\tau_k)\Psi(t-z,\tau_k)\,dt=\int_0^{\infty}\psi(t,\tau_k)(\Psi(t-z,\tau_k)+\Psi(t+z,\tau_k))\,dt.$$

By means of the relations [7]

$$\frac{1}{\pi}\int_0^{\infty}\frac{\exp(-y)}{\sqrt{y}}\,K_0(|x-y|)L_m^{(-1/2)}(2y)\,dy=\sqrt{\frac{\pi}{2}}\,\frac{(2m-1)!!}{(2m)!!}\,\exp\left(-x\right)L_m^{(-1/2)}(2x)$$

$$((-1)!!=0!!=1, \quad (2m-1)!!=1\cdot3\cdots(2m-1), \quad (2m)!!=2\cdot4\cdots(2m)),$$

the integrals with a logarithmic singularity can be calculated exactly, and the regular integrals can be found approximately using the quadrature formula [8]

$$\int_0^\infty \frac{\exp(-x)}{\sqrt{x}} f(x)\,dx = \sum_{i=1}^{M+1} w_i f(x_i),$$

where

$$w_i = \frac{\sqrt{\pi}(2M+1)!!\,x_i}{(2M+2)!!((M+2)L_{M+2}^{(-1/2)}(x_i))^2}$$

are the weighting coefficients and x_i are the zeroes of the Laguerre polynomial $L_{M+1}^{(-1/2)}(x)$, $i = 1, \ldots, M+1$.

To find the unknown coefficients $B_m(\tau_k)$ and $C_{j,m}(\tau_k)$, we apply, at each time τ_k, the collocation method [9], selecting a set of (generally not equidistant) points z_m ($m = 0, \ldots, M$). Then, the system reduces to a system of linear algebraic equations of dimension $[3(M+1) \times 3(M+1)]$ for the unknown coefficients that define the distribution of the contact pressure and the variation of the functions f_j along the z axis at that time. Here, the following remarks have to be made.

Remark 1. The expansion coefficients are calculated in several stages.

First, we consider only system (9) of integral equations on the chosen set of points. We solve the resultant system of linear algebraic equations, find the distribution of the contact pressure, and check if the conditions $p(z_m, \tau_k) \geqslant 0$ are fulfilled.

Then, at the points z_m where the contact pressure is negative, we satisfy conditions (11) or conditions of type (10) with h_c substituted by h_b. The beginning of the unloaded contact zone (point z_{m_0}) can be found from the condition $|p(z_m, \tau_k)| < \varepsilon$ ($\varepsilon \approx 10^{-5}$) for all values $m_0 \leqslant m \leqslant M$. At the points z_m of this zone, conditions (10) are satisfied. This procedure is repeated until the contact pressure reverses its sign.

Remark 2. The possibility of neglecting the solution of the system over the interval $[z_M, \infty)$ is based on the fact that the contact pressure and temperature vanish rather rapidly [1]. At a distance of $10L$ (L is the parameter that defines the interval of application of the uniformly distributed load), these quantities differ little from zero. It should be noted that discontinuous functions $f_j(z, \tau_k)$ are approximated here; for this reason, the number of division points z_m and, hence, the number of terms in expansions (12) should be rather large. As the calculations showed, for $M = 200$ and a uniform grid chosen over the interval $z \in [0, 2L]$ with a step $L/40$, the relative calculation error is within 3%. The time step was chosen equal to $\tau_1 = 1$ sec, and a uniform grid with a step $L/15$ was chosen over the interval $(2L, 10L)$. The calculation accuracy was verified by reducing the intervals over the time and the z coordinate, and also by doubling the integration interval. Sums (12) were calculated by the Fejér method [5].

Analysis of Results and Conclusions. The problem was numerically analyzed for the "steel–steel" friction pair ($E_j = 2 \cdot 10^5$ MPa, $\nu_j = 0.3$, $\lambda_j = 50$ W/(m $\cdot$ K), and $k_j = 0.125 \cdot 10^{-4}$ m^2/sec) with the following values of the governing parameters: $h_a = 10$ kW/(m$^2 \cdot$ K), $f = 0.1$, $\gamma_1 = \gamma_2 = 20$ m^{-1}, $a_1 = 3.5$ cm, $a_0 = 5$ cm, $a_2 = 6$ cm, $\alpha_1 = (1\text{–}15) \cdot 10^{-6}$ K^{-1}, and $\alpha_2 = 12 \cdot 10^{-6}$ K^{-1}. In this case, conditions (11) are satisfied in the separation zone, and the thermal conductivity coefficient in the loaded contact zone is $h_c = 0.5 h_a$. The load applied to the surface $r = a_2$ and the relative angular velocity of revolution vary according to the laws

$$q_2(z, \tau) = q_{\rm st}(z)(1 - \exp(-\beta\tau)), \qquad \omega(\tau) = \omega_0;$$

$$q_2(z, \tau) = q_{\rm st}(z), \qquad \omega(\tau) = \omega_0(1 - \exp(-\beta\tau)),$$

where $q_{\rm st}(z) = q_0 H(L - |z|)$. Here, $q_0 = 20$ MPa, $\omega_0 = 0$–2 rad/sec, $\beta = 0.01$ sec^{-1}, and $L = 0.1$ m. As was noted above, the surface $r = a_1$ is free of the load.

Below, we report some results of a numerical analysis of the problem. Figure 1 shows the distribution of the stationary contact pressure. Depending on the relation between the linear thermal expansion coefficients of the cylinders, three interaction mechanisms are possible:

— If $\alpha_1 \ll \alpha_2$, then the cylinders contact each other in a restricted region in the case of a simply connected region of loading;

— If $\alpha_1 \approx \alpha_2$, then the loaded contact region is multiply connected;

— If $\alpha_1 > \alpha_2$, then the cylinders contact each other over their entire surface area.

It should be noted here that an increase in heat-release intensity due to increasing ω_0 causes a decrease in the contact area and in the contact pressure if $\alpha_1 < \alpha_2$. Alternatively, if $\alpha_1 > \alpha_2$, then an increase in ω_0 results in an increase in $p_{\rm st}(z)$. The range of $\omega_0 \geqslant 0$ is bounded by some critical value $\omega_{\rm cr}$ [1].

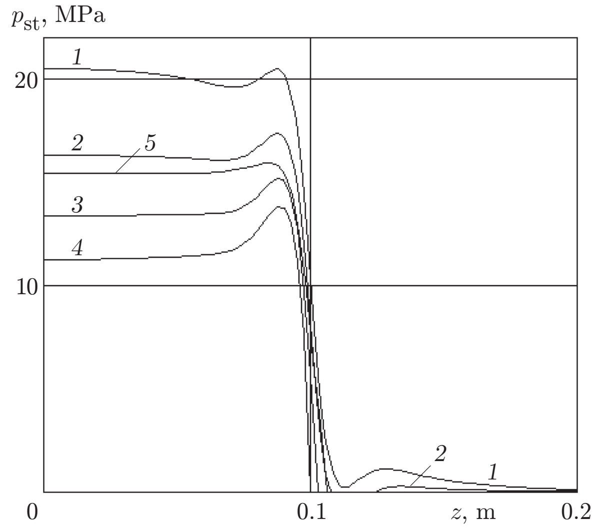
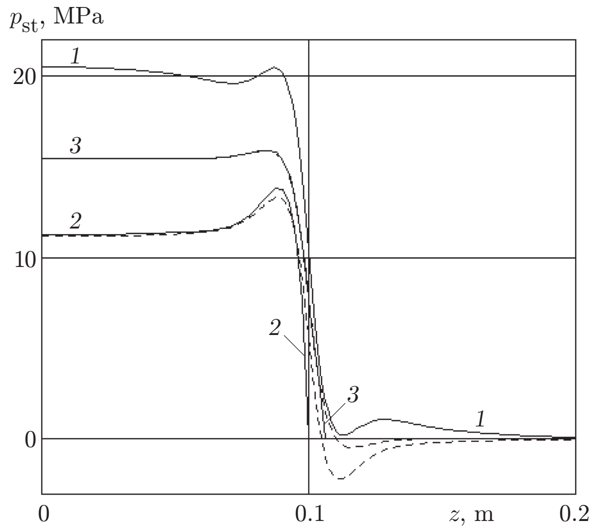


Fig. 1. Distributions of stationary contact pressure for different values of α_1 ($\alpha_2 = 12 \cdot 10^{-6}$ K^{-1} and $\omega_0 = 1$ rad/sec): $\alpha_1 = 15 \cdot 10^{-6}$ (1), $12 \cdot 10^{-6}$ (2), $9 \cdot 10^{-6}$ (3), and $6 \cdot 10^{-6}$ K^{-1} (4); curve 5 refer to the contact pressure in elastic interaction ($\omega_0 = 0$).



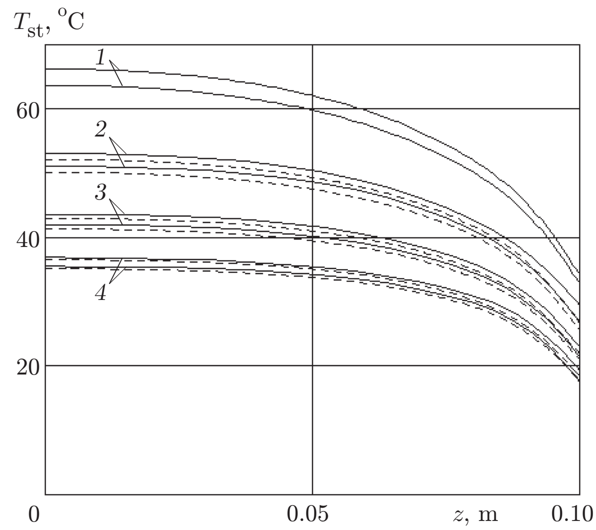


Fig. 2

Fig. 3

Fig. 2. Distributions of stationary contact pressure in the approximate (dashed curves) and refined (solid curves) formulations of the problem: curves 1 and 2 refer to $\alpha_1 = 15 \cdot 10^{-6}$ and $6 \cdot 10^{-6}$ K^{-1}, respectively; curves 3 refer to the contact pressure during elastic interaction.

Fig. 3. Distributions of stationary temperature on the surface $r = a_0$ for $\alpha_1 = 15 \cdot 10^{-6}$ (1), $12 \cdot 10^{-6}$ (2), $9 \cdot 10^{-6}$ (3), and $6 \cdot 10^{-6}$ K^{-1} (4); the solid curves show the refined formulation of the problem; the dashed curves refer to the approximate formulation of the problem.

For comparison, the dashed curves in Fig. 2 show the distributions of contact pressure calculated under the assumption that the cylinder are in intimate contact [1]. As it could be expected, the curves of contact pressure obtained for $\alpha_1 = 15 \cdot 10^{-6}$ K^{-1} in the approximate and refined (with allowance for separation) formulations coincide. An insignificant difference between the contact-pressure distributions obtained for the elastic interaction in the approximate and refined formulations can be attributed to the fact that the tribosystem here is self-balanced, i.e., unlike classical contact problems, there is no integral condition of equality of sums of contact stresses to the

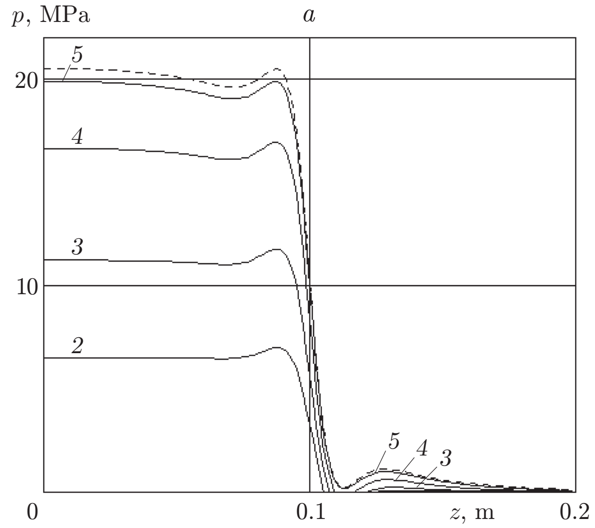


Fig. 4. Distributions of pressure in the stationary formulation of the problem (dashed curves) and distributions of nonstationary contact pressure (solid curves) under varied load (a) and under varied angular velocity of revolution (b): $\tau = 0$ (1), 50 (2), 100 (3), 200 (4), and 400 sec (4) ($\alpha_1 = 15 \cdot 10^{-6}$ K^{-1}, $\alpha_2 = 12 \cdot 10^{-6}$ K^{-1}, and $\omega_0 = 1$ rad/sec).

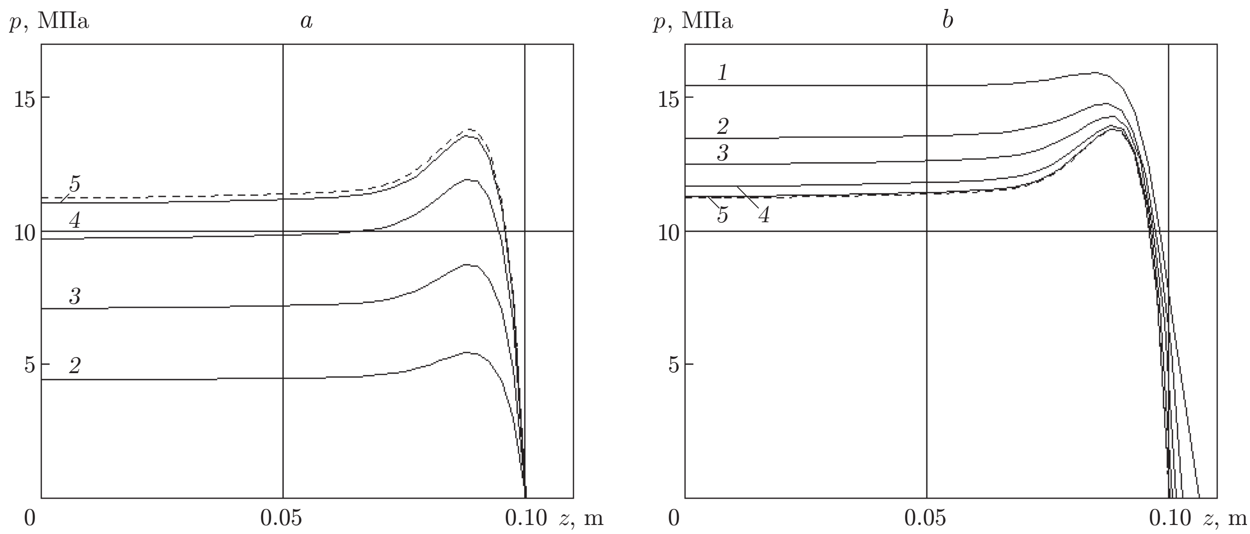


Fig. 5. Distributions of pressure in the stationary formulation of the problem (dashed curves) and distributions of nonstationary contact pressure (solid curves) at various times τ for $\alpha_1 = 6 \cdot 10^{-6}$ K^{-1} and $\alpha_2 = 12 \cdot 10^{-6}$ K^{-1} (notation the same as in Fig. 4.)

pressing force in the problem under consideration. The difference between the distributions $p_{\rm st}(z)$ obtained in the approximate and refined formulations for $\alpha_1 = 6 \cdot 10^{-6}$ K^{-1} can be attributed to the variation of thermal boundary conditions in the separation zone.

The difference in the thermal boundary conditions in the separation zone is also responsible for the behavior of temperature distributions on the surface $r = a_0$ in the approximate and refined formulations (Fig. 3). Curves 1–4 in Fig. 3 are plotted for the conditions of Fig. 1; the upper and lower curve in each group refer to the inner and outer cylinders, respectively.

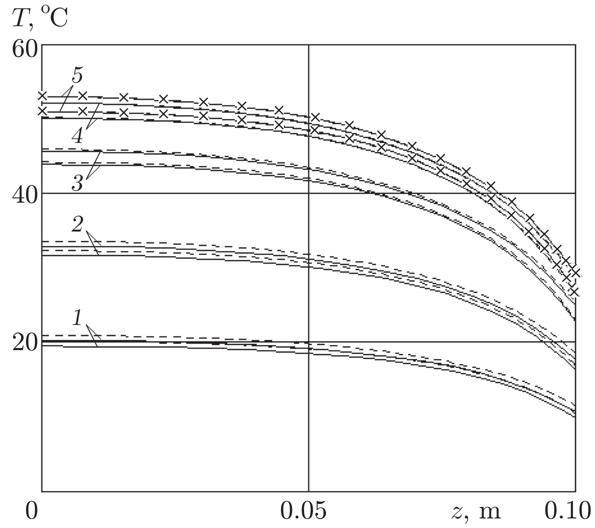


Fig. 6. Distributions of stationary temperature (curves drawn through crosses) and distributions of nonstationary temperature for $\tau = 50$ (1), 100 (2), 200 (3), 400 (4), and 600 sec (5) ($\alpha_1 = \alpha_2 = 12 \cdot 10^{-6}$ K^{-1}); the solid and dashed curves refer to for time-dependent ω and load, respectively.

An examination of the solution of the quasi-static problem shows that, provided that the condition $\omega_0 < \omega_{\rm cr}$ is fulfilled, the contact stress monotonically reaches the corresponding stationary value, and its behavior is determined by the chosen time dependences of the load and angular velocity of revolution. In the first case (time-dependent load), the contact area (for $\alpha_1 < \alpha_2$) remains unchanged (Fig. 4a), whereas in the second case it monotonically decreases (Fig. 4b). For $\alpha_1 \gg \alpha_2$, the effect of multiple connectedness of the contact region is manifested as the contact stresses reach the steady value (complete contact).

Figures 4 and 5 shows the distributions of contact pressure during nonstationary heat release caused by the variation of the load and angular velocity of revolution in time. In both cases, the load and the angular velocity of revolution, whose values reach a stationary value during the time $\tau = 450$ sec, determine the duration of the transient process for contact stresses of the order of 600–700 sec.

The temperature of the contact surface reaches a stationary value more slowly (approximately in 800 sec), and the duration of the transient process increases with distance from the surface $r = a_0$. Figure 6 shows the distributions of temperature over the surface $r = a_0$ for each cylinder, which were obtained that the cylinders have identical linear thermal expansion coefficients. The upper and lower curves in each group refer to $j = 1$ and $j = 2$, respectively.

Generally speaking, in problems where separation is possible, conditions of an imperfect thermal contact should be set at the interface between the layers. In this case, in view of complexity of determination of the thermal conductivity coefficients in each zone, it is reasonable to introduce one averaged coefficient. In other words, mixed mechanical conditions are set at the interface, and one thermal condition is specified over the entire surface. A numerical analysis of the problem performed for this case showed that, provided that the averaged thermal conductivity coefficient of the contact surface coincides with the thermal conductivity coefficient of the loaded contact zone, the contact stresses differ from the stresses obtained under mixed thermal conditions within 1%, and the character of the difference between the thermoelastic contact stresses obtained in the approximate and refined formulations is similar to the case of elastic interaction (curve 3 in Fig. 2).

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